Information dynamics in cavity QED

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Abstract

A common experimental setup in cavity quantum electrodynamics (QED) consists of a single two-level atom interacting with a single mode of the electromagnetic field inside an optical cavity. The cavity is externally driven and the output is continuously monitored via homodyne measurements. We derive formulas for the optimal rates at which these measurements provide information about (i) the quantum state of the system composed of atom and electromagnetic field, and (ii) the coupling strength between atom and field. We find that the two information rates are anticorrelated.

I. INTRODUCTION

In this paper, we consider a system consisting of a single two-level atom, located inside an externally driven optical cavity. The atom interacts with a single light mode inside the cavity, which is coupled to the environment via a partially transparent mirror. The output field from the cavity is monitored using the homodyne detection scheme, in which the cavity output is added to a reference field at a beam-splitter and then analyzed by two photodetectors (see Fig. 1).

In two recent ground-breaking experiments [1, 2], a similar setup was used to observe the trajectory of the atom inside the cavity. In these experiments, the atomic position was inferred from the strength of the atom-cavity coupling, which can be estimated directly from the photocurrents [3]. To characterize the performance of this *atom-cavity microscope*, Kimble introduced a quantity called *optical information* [4, 5], which measures the rate at which the measurement provides information about the system. In Refs. [4, 5], however, no formal definition of optical information is given, and only a heuristic derivation of its value is provided. Recently, a number of related definitions of information gain have been proposed and investigated numerically for a simpler quantum-optical system [6].

The question of how much information about a monitored system is provided by a continuous measurement is interesting in its own right. In this paper, we consider two, as it turns out, complementary cases. In Sec. III A we calculate the optimal rate at which a homodyne measurement provides information about the quantum state of the system composed of the electromagnetic field and the internal state of the atom. In Sec. III B, we calculate the optimal rate at which the measurement gives information about the coupling strength between the atom and the intra-cavity field. In Sec. II, we introduce our mathematical model and main assumptions, and Sec. IV concludes the paper with a short discussion.

II. MODEL

The evolution of the state, ρ, of an open quantum system subject to a continuous measurement can often be described by a stochastic master equation of the form [7, 8, 9]

$$d\rho = \mathcal{L}(\rho)dt + \mathcal{M}(\rho)dW\,, \tag{1}$$

which is understood in the sense of the Itô stochastic differential calculus [10]. Any particular measurement record is represented by some realization of the stochastic process $W(t)$. The superoperator $\mathcal{L}$ is linear and defines the "unconditional" evolution, i.e., the evolution in the absence of measurements. By contrast, the superoperator $\mathcal{M}$ is nonlinear and accounts for the effects of the measurement.

The cavity-QED system we are considering here is illustrated in Fig. 1. A single two-level atom, with ground state $|\mathrm{g}\rangle$ and excited state $|\mathrm{e}\rangle$ is located inside a high-finesse optical cavity, which is driven by an external laser field of strength E. The atom interacts with a single light mode of the cavity. We denote by g the strength of the atom-cavity coupling, and by κ the cavity field decay rate. In this paper, we assume that the atom, the cavity and the driving laser are all resonant.

In the absence of measurements, $\mathcal{M} = 0$ in Eq. (1), and the joint density operator of atom and intra-cavity field, ρ, obeys the master equation

$$\dot{\rho} = \mathcal{L}(\rho) \; , \tag{2}$$

where

$$\mathcal{L}(\rho) = \big[E(a^\dagger - a) + g(a^\dagger \sigma - \sigma^\dagger a), \rho\,\big] + \kappa(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) \, . \tag{3}$$

Here a is the annihilation operator for the cavity field mode, and $\sigma = |\mathrm{g}\rangle\langle\mathrm{e}|$. In the following we focus our attention on the important case of the strong driving regime ($E/g \gg 1$). In this regime the system approaches a steady state of the form [11]

$$\rho_{\mathrm{ss}}^{\alpha} = \frac{1}{2}\big(|\alpha;+\rangle\langle\alpha;+| + |\alpha^*;-\rangle\langle\alpha^*;-|\big) \; . \tag{4}$$

Here $|\alpha;+\rangle$ and $|\alpha^*;-\rangle$ are two orthogonal quantum states defined by

$$\begin{aligned} |\alpha;+\rangle &= \frac{1}{\sqrt{2}}|\alpha\rangle\big(|\mathrm{g}\rangle + i|\mathrm{e}\rangle\big) \, , \\ |\alpha^*;-\rangle &= \frac{1}{\sqrt{2}}|\alpha^*\rangle\big(|\mathrm{g}\rangle - i|\mathrm{e}\rangle\big) \, , \end{aligned} \tag{5}$$

where $|\alpha\rangle$ is the coherent field state with amplitude

$$\alpha = \frac{E}{\kappa}\left[1 - \left(\frac{g}{2E}\right)^2 + i\frac{g}{2E}\sqrt{1 - \left(\frac{g}{2E}\right)^2}\;\right] \; . \tag{6}$$

Let us now assume that the field escaping from our system undergoes a continuous measurement via the standard homodyne measurement scheme [12]. In this scheme we have

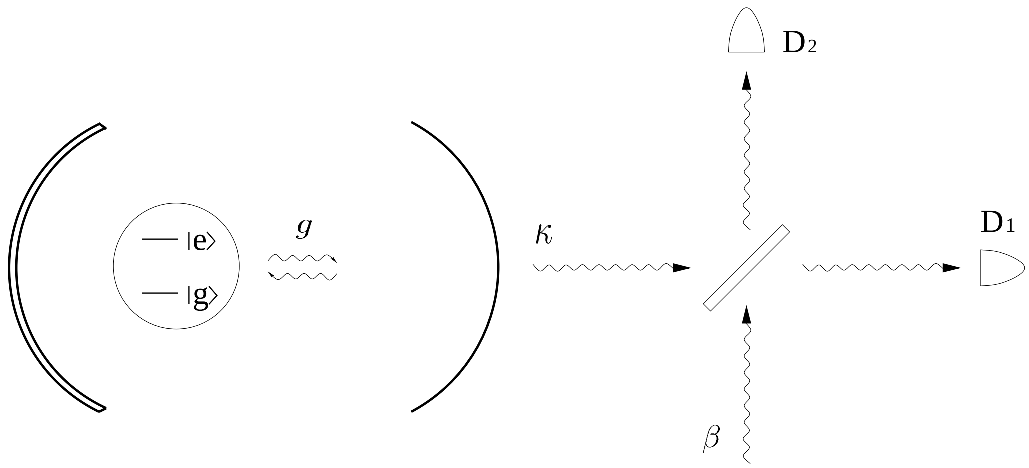

FIG. 1: Homodyne or heterodyne measurements in cavity QED. See the text for an explanation of the parameters.

only one free complex parameter: the reference field β that is added to the cavity output at a beam splitter prior to the detection by two detectors D_1 and D_2 (see Fig. 1). If the measurement record consists of the scaled difference photocurrent $dq/dt \equiv (I_2 - I_1)/|\beta|$, where I_1 and I_2 are the photocurrents detected by D_1 and D_2, respectively, then the measurement operator $\mathcal{M}$ in the stochastic master equation (1) is given by [8, 9]

$$\mathcal{M}(\rho) = \sqrt{2\kappa\eta}\left(e^{-i\phi}a\rho + e^{i\phi}\rho a^\dagger - \mathrm{tr}\big[\rho(e^{-i\phi}a + e^{i\phi}a^\dagger)\big]\,\rho\right)\,, \tag{7}$$

where η is the efficiency of the photodetection and $\phi = \arg\beta$ is the phase of the reference field. The realization of the Wiener process $W(t)$ in Eq. (1) is connected to the experimentally observed difference photocurrent via the relation [8, 9]

$$dq = 2\kappa\eta\,\mathrm{tr}\big[\rho(e^{i\phi}a^\dagger + e^{-i\phi}a)\big]\,dt + \sqrt{2\kappa\eta}\,dW\,. \tag{8}$$

In the special case $\rho = \rho_{\rm ss}^{\alpha}$ this equation becomes

$$dq = 4\kappa\eta\,\mathrm{Re}(\alpha)\cos\phi\,dt + \sqrt{2\kappa\eta}\,dW\,. \tag{9}$$

III. INFERENCE

A. Information about the quantum state

The amount of information provided by the measurement about the quantum state, ρ, is quantified by the reduction of the von Neumann entropy,

$$H(\rho) \equiv -\mathrm{tr}(\rho \ln \rho) . \tag{10}$$

In this subsection we calculate the rate at which this entropy changes as a result of measurements. The average rate of entropy change in the presence of continuous observations is given by

$$\langle \dot{H}(\rho) \rangle = \lim_{\Delta t \to 0} \left\langle \frac{H(\rho + \delta) - H(\rho)}{\Delta t} \right\rangle \tag{11}$$

where the average $\langle \cdot \rangle$ is taken over all possible measurement outcomes observed over the time Δt, and

$$\delta = \mathcal{L}(\rho)\Delta t + \mathcal{M}(\rho)\Delta W . \tag{12}$$

We now assume that the system is initially in a steady state, ρ_{ss}, defined by the relation

$$\mathcal{L}(\rho_{\mathrm{ss}}) = 0 . \tag{13}$$

For simplicity, we furthermore assume that

$$[\mathcal{M}(\rho_{\mathrm{ss}}), \rho_{\mathrm{ss}}] = 0 . \tag{14}$$

This last assumption is satisfied by Eqs. (4) and (7). At the end of this section we will outline a more general framework which is free from these assumptions. Since the stochastic master equation (1) preserves the positivity of the density operator, ρ, the support $\mathcal{M}(\rho_{\mathrm{ss}})$ must lie in the support of ρ_{ss}, $\mathrm{supp}(\mathcal{M}(\rho_{\mathrm{ss}})) \subseteq \mathrm{supp}(\rho_{\mathrm{ss}})$. Now consider a basis in which ρ_{ss} and $\mathcal{M}(\rho_{\mathrm{ss}})$ are both diagonal. In this basis, let $a_1, \ldots, a_m$ be the nonzero diagonal elements of ρ_{ss} and let $b_1, \ldots, b_m$ be the corresponding diagonal elements of $\mathcal{M}(\rho_{\mathrm{ss}})$.

Substituting $\rho = \rho_{\mathrm{ss}}$ into Eq. (11), we obtain

$$\begin{aligned} \langle \dot{H}(\rho_{\mathrm{ss}}) \rangle &= \lim_{\Delta t \to 0} \left\langle \frac{H(\rho_{\mathrm{ss}} + \mathcal{M}(\rho_{\mathrm{ss}})\Delta W) - H(\rho_{\mathrm{ss}})}{\Delta t} \right\rangle \\ &= \lim_{\Delta t \to 0} \left\langle \frac{-\sum_{k=1}^{m} (a_k + b_k \Delta W) \ln(a_k + b_k \Delta W) + \sum_{k=1}^{m} a_k \ln a_k}{\Delta t} \right\rangle \end{aligned}$$

$$= -\sum_{k=1}^{m} \frac{(b_k)^2}{2a_k}\,, \tag{15}$$

where we have expanded the logarithm to second order in ΔW and used the relation $\langle(\Delta W)^2\rangle = \Delta t$.

For any state $\rho_{\rm ss}^{\alpha}$ of the form (4) we have

$$\mathcal{M}(\rho_{\rm ss}^{\alpha}) = \sin\phi\,\sqrt{2\kappa\eta}\,\mathrm{Im}(\alpha)\Big(|\alpha;+\rangle\langle\alpha;+| - |\alpha^*;-\rangle\langle\alpha^*;-|\Big)\,. \tag{16}$$

Since $\rho_{\rm ss}^{\alpha}$ and $\mathcal{M}(\rho_{\rm ss}^{\alpha})$ commute, we can apply Eq. (15). For the particular value for α given by Eq. (6), the diagonal elements of $\rho_{\rm ss}^{\alpha}$ and $\mathcal{M}(\rho_{\rm ss}^{\alpha})$ are

$$a_{1,2} = 1/2\,, \quad \text{and} \quad b_{1,2} = \pm g\,\sin\phi\sqrt{\frac{\eta}{2\kappa}\Big(1-(\frac{g}{2E})^2\Big)}\,. \tag{17}$$

Finally, we obtain the main result of this subsection: the rate of information gain about the quantum state of the system is

$$R_Q \equiv -\langle\dot{H}(\rho_{\rm ss}^{\alpha})\rangle = \frac{g^2\eta}{\kappa}\Big(1-\big(\frac{g}{2E}\big)^2\Big)\sin^2\phi\,. \tag{18}$$

Here the minus sign indicates that the information gain corresponds to the reduction of uncertainty about the system state which is measured by the von Neumann entropy.

We now finish this subsection by developing some general formalism that makes no assumptions regarding the superoperators $\mathcal{L}$ and $\mathcal{M}$. Equation (10) can be expanded in the form

$$\begin{aligned} H(\rho) &= \mathrm{tr}\Big[\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\rho(\rho-\mathbb{1})^n\Big] \\ &= \mathrm{tr}\Big[\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\tilde{\rho}^{n+1}+\tilde{\rho}^n)\Big]\,, \end{aligned} \tag{19}$$

where $\tilde{\rho} = \rho - \mathbb{1}$. Equation (11) then becomes

$$\langle\dot{H}(\rho)\rangle = \lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,\Delta t}\left\langle \mathrm{tr}[(\tilde{\rho}+\delta)^{n+1}-\tilde{\rho}^{n+1}] + \mathrm{tr}[(\tilde{\rho}+\delta)^n-\tilde{\rho}^n]\right\rangle\,. \tag{20}$$

Keeping the terms to the second order in δ, we have

$$(\rho+\delta)^n = \rho^n + \sum_{l=0}^{n-1}\rho^l\,\delta\,\rho^{n-1-l} + \sum_{p=0}^{n-2}\sum_{q=0}^{n-2-p}\rho^p\,\delta\,\rho^{n-2-(p+q)}\delta\,\rho^q + O(\delta^3)\,. \tag{21}$$

Using the cyclic property of the trace,

$$\mathrm{tr}[(\rho+\delta)^n - \rho^n] \;=\; \mathrm{tr}\Big[n\rho^{n-1}\,\delta + \sum_{p=0}^{n-2}\sum_{q=0}^{n-2-p} \rho^{p+q}\;\delta\;\rho^{n-2-(p+q)}\delta + O(\delta^3)\Big]$$
$$\;=\; \mathrm{tr}\Big[n\rho^{n-1}\,\delta + \sum_{s=0}^{n-2}(s+1)\rho^{s}\;\delta\;\rho^{n-2-s}\delta + O(\delta^3)\Big]\,. \tag{22}$$

Combining Eqs. (20) and (22), we obtain

$$\langle \dot{H}(\rho)\rangle = \lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,\Delta t}\mathrm{tr}\Big\langle\,(n+1)\tilde{\rho}^n\delta + \sum_{s=0}^{n-1}(s+1)\tilde{\rho}^s\delta\tilde{\rho}^{n-1-s}\delta$$
$$+ n\tilde{\rho}^{n-1}\delta + \sum_{s=0}^{n-2}(s+1)\tilde{\rho}^s\delta\tilde{\rho}^{n-2-s}\delta\,\Big\rangle\,. \tag{23}$$

Substituting Eq. (12) and using $\langle \Delta W\rangle = 0$, $\langle (\Delta W)^2\rangle = \Delta t$ we finally arrive at

$$\langle \dot{H}(\rho)\rangle = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{tr}\Big(\,[(n+1)\tilde{\rho}^n + n\tilde{\rho}^{n-1}]\mathcal{L}(\rho)$$
$$+\sum_{s=0}^{n-1}(s+1)\tilde{\rho}^s\mathcal{M}(\rho)\tilde{\rho}^{n-1-s}\mathcal{M}(\rho)$$
$$+\sum_{s=0}^{n-2}(s+1)\tilde{\rho}^s\mathcal{M}(\rho)\tilde{\rho}^{n-2-s}\mathcal{M}(\rho)\,\Big)\,. \tag{24}$$

This formula is valid without any restrictions on the superoperators $\mathcal{L}$ and $\mathcal{M}$. It is straightforward to show that, under the previous assumptions $\rho = \rho_{\rm ss}$ and $[\mathcal{M}(\rho_{\rm ss}), \rho_{\rm ss}] = 0$, Eq. (24) reduces to Eq. (15) as required.

B. Information about the atom-cavity coupling

Since the quantized degrees of freedom described by the density operator ρ of the previous sections do not include the atomic position, a different approach is needed to obtain information about the atom's trajectory inside the cavity. The key is the fact that the atom-cavity coupling g depends on the position of the atom in a known way (see, e.g., Ref. [4]). In this subsection, we will therefore focus on calculating the rate at which the measurement provides information about the parameter g.

Let q be a photocharge obtained by integrating the difference photocurrent, given by Eq. (8), over a small time interval Δt. Introduce a small parameter ϵ such that $\kappa\Delta t \sim \epsilon^{3/2}$

and $|\beta| \sim \epsilon^{-1}$. The existence of such ϵ is assumed in the standard derivation of Eqs. (7) and (8) [8, 9]. Equation (8) is derived in the limit of small ϵ when q can be treated as a Gaussian random variable,

$$G(q, \langle q \rangle, v^2) \equiv \frac{1}{\sqrt{2\pi v^2}} \exp\Big(- \frac{(q - \langle q \rangle)^2}{2v^2} \Big) , \tag{25}$$

with mean

$$\langle q \rangle = 2\kappa\eta \,\mathrm{tr}[\rho(e^{i\phi}a^\dagger + e^{-i\phi}a)]\Delta t + O(\epsilon^2) \tag{26}$$

and variance

$$v^2 = 2\kappa\eta\Delta t + O(\epsilon^2) . \tag{27}$$

So far we have treated the atom-cavity coupling g as a known parameter. This means that the above equations give us the conditional probability density $P(q|g)$ of registering measurement result q given that the atom-cavity coupling is equal to g,

$$P(q|g) = G(q, \langle q \rangle, v^2). \tag{28}$$

The conditional probability of the atom-cavity coupling to be equal to g given a particular measurement result q can be derived from the Bayes rule,

$$P(g|q) = \frac{P(q|g)P(g)}{\int P(q|g)P(g)dg} , \tag{29}$$

where $P(g)$ is a probability distribution that characterizes our knowledge of g prior to obtaining q.

To quantify the information gain, we use the entropy S of a continuous probability distribution $f(x)$ defined by

$$S[f(x)] \equiv - \int f(x) \ln f(x)\, dx . \tag{30}$$

The average rate at which observation of q gives us information about g is then given by

$$R_g \equiv \lim_{\Delta t \to 0} \Big\langle \frac{-\Delta S}{\Delta t} \Big\rangle = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (-\Delta S) P(q|g)\, dq , \tag{31}$$

where

$$\Delta S = S[P(g|q)] - S[P(g)] . \tag{32}$$

On the relatively slow time scales of the atomic motion one can assume to a very good approximation that the quantized degrees of freedom described by the density matrix ρ are in the steady state given by Eq. (4). In this case Eq. (26) gives

$$(q - \langle q \rangle)^2 \;=\; q^2 - 4q\kappa\eta \,\mathrm{tr}\big[\rho(e^{i\phi}a^\dagger + e^{-i\phi}a)\big]\Delta t + O(q\epsilon^2) + O(\epsilon^3)$$

$$= q^2 - 8qE\eta\cos\phi\left(1-(\frac{g}{2E})^2\right)\Delta t + O(q\epsilon^2) + O(\epsilon^3)\ . \qquad (33)$$

We can now derive a simple formula for R_g if we make the further convenient assumption that the prior distribution $P(g)$ is a Gaussian with variance v_0^2,

$$P(g) = G(g, \langle g\rangle, v_0^2)\ . \qquad (34)$$

The posterior probability density $P(g|q)$ will then also be a Gaussian,

$$P(g|q) = G(g, m, v_1^2)\ , \qquad (35)$$

where the variance v_1^2 is given by

$$v_1^2 = \frac{v_0^2\kappa E}{\kappa E + q v_0^2\cos\phi}\ . \qquad (36)$$

This expression, where we neglected the $O(q\epsilon^2)$ and $O(\epsilon^3)$ terms in Eq. (33), is valid for small q. The value of the mean m is irrelevant for the current argument. By a direct calculation, we obtain the entropy change

$$\Delta S = -\frac{1}{2}\ln(v_0^2/v_1^2) = -\frac{1}{2}\ln\left(1 + q\frac{\cos\phi}{\kappa E}v_0^2\right) \simeq -q\frac{\cos\phi}{2\kappa E}v_0^2\ . \qquad (37)$$

Due to the limit $\Delta t\to 0$, the Gaussian $P(q|g)$ in Eq. (31) is strongly peaked at small values of q. Hence we can substitute the above estimation of ΔS into Eq. (31). We thus obtain our final result, the average information rate

$$R_g = \frac{2v_0^2\eta}{\kappa}\left(1-(\frac{g}{2E})^2\right)\cos^2\phi\ . \qquad (38)$$

IV. DISCUSSION

Comparing the expressions Eq. (18) for the information rate R_Q and Eq. (38) for the information rate R_g, one notices their very similar structure. The rate R_Q, referring to information about the quantum state of the atom-cavity system, is proportional to $\sin^2\phi$, whereas the rate R_g, referring to information about the parameter g, is proportional to $\cos^2\phi$. In both formulas, ϕ is the phase of the reference field β. The proportionality factor in R_g is obtained from the proportionality factor in R_Q by substituting for g^2 twice the prior variance, $2v_0^2$, of the random variable g.

This means that there is a trade-off between the two information rates: the more we learn about the coupling parameter (and hence the atomic position), the less we can learn about the atom-cavity quantum state and vice versa. This is a manifestation of the uncertainty principle applied to the conjugate field quadratures $X = a + a^\dagger$ and $Y = i(a - a^\dagger)$. Let us elaborate on this point by considering our starting equations, i.e. Eq. (8), which gives our measurement record, and Eqs. (1), (3), and (7) for the corresponding change of the system density matrix. As before, we assume the steady state $\rho_{\rm ss}$ as the initial condition for these equations. The qualitative conclusions of this discussion do not depend on any specific prior distribution for g.

We notice that the average photocurrent is proportional to the expectation value of the expression $e^{i\phi}a^\dagger + e^{-i\phi}a$. In this sense the cases $\phi = 0$ and $\phi = \pi/2$ correspond to measuring the expectation values of X and Y quadratures respectively. In the case of $\phi = 0$ the nonlinear contribution from the measurement, given by Eq. (7), becomes proportional to $\rho_{\rm ss}^{\alpha}$. From Eq. (1) we therefore see that the system density matrix remains equal to $\rho_{\rm ss}^{\alpha}$ at all times, independent of the measurement record. This means that for $\phi = 0$ the measurement record contains no information about the quantum state of the system, in agreement with (18). At the same time the setting $\phi = 0$ maximizes the average photocurrent, which carries information about g [see Eq. (9)]. The second stochastic term in (9) does not depend on g and so, from the point of view of learning about g, this stochastic term is nothing but noise superimposed on the average photocurrent. We therefore see that maximizing the average photocurrent increases the signal-to-noise ratio for measuring g and thus $\phi = 0$ gives us the maximum information about g, in agreement with Eq. (38). In summary, the case $\phi = 0$ is ideal for measuring g and it gives no information about the internal state. In the case $\phi = \pi/2$ we have the opposite situation: we obtain maximum information about the atom-cavity state and no information about g, because the measured photocurrent becomes independent of g. For a numerical comparison of the two information rates for a simpler quantum-optical system and a wide range of detection schemes, see Ref. [6].

At first sight, learning about the quantum state of the system seems to be a very different problem form learning about the atomic position. It so happens that within our framework these two tasks are best accomplished by monitoring conjugate observables, namely the X and Y field quadratures. With a straightforward modification our calculations can be also applied to the heterodyne detection scheme. For heterodyne detection, both field quadratures are monitored simultaneously, at the cost of reducing the respective maximal information rates.

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